

# A New Outer Bound and the Noisy-Interference Sum-Rate Capacity for Gaussian Interference Channels

Xiaohu Shang, Gerhard Kramer, and Biao Chen

## Abstract

A new outer bound on the capacity region of Gaussian interference channels is developed. The bound combines and improves existing genie-aided methods and is shown to give the sum-rate capacity for *noisy interference* as defined in this paper. Specifically, it is shown that if the channel coefficients and power constraints satisfy a simple condition then single-user detection at each receiver is sum-rate optimal, i.e., treating the interference as noise incurs no loss in performance. This is the first concrete (finite signal-to-noise ratio) capacity result for the Gaussian interference channel with weak to moderate interference. Furthermore, for certain mixed (weak and strong) interference scenarios, the new outer bounds give a corner point of the capacity region.

*Index terms* — capacity, Gaussian noise, interference.

## I. INTRODUCTION

The interference channel (IC) models communication systems where transmitters communicate with their respective receivers while causing interference to all other receivers. For a two-user Gaussian IC, the channel output can be written in the standard form [1]

$$\begin{aligned} Y_1 &= X_1 + \sqrt{a}X_2 + Z_1, \\ Y_2 &= \sqrt{b}X_1 + X_2 + Z_2, \end{aligned}$$

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where  $\sqrt{a}$  and  $\sqrt{b}$  are channel coefficients,  $X_i$  and  $Y_i$  are the transmit and receive signals, and where the user/channel input sequence  $X_{i1}, X_{i2}, \dots, X_{in}$  is subject to the power constraint  $\sum_{j=1}^n \mathcal{E}(X_{ij}^2) \leq nP_i$ ,  $i = 1, 2$ . The transmitted signals  $X_1$  and  $X_2$  are statistically independent. The channel noises  $Z_1$  and  $Z_2$  are possibly correlated unit variance Gaussian random variables, and  $(Z_1, Z_2)$  is statistically independent of  $(X_1, X_2)$ . In the following, we denote this Gaussian IC as  $\text{IC}(a, b, P_1, P_2)$ .

The capacity region of an IC is defined as the closure of the set of rate pairs  $(R_1, R_2)$  for which both receivers can decode their own messages with arbitrarily small positive error probability. The capacity region of a Gaussian IC is known only for three cases:

- $a = 0, b = 0$ .
- $a \geq 1, b \geq 1$ : see [2]–[4].
- $a = 0, b \geq 1$ ; or  $a \geq 1, b = 0$ : see [5]

For the second case both receivers can decode the messages of both transmitters. Thus this IC acts as two multiple access channels (MACs), and the capacity region for the IC is the intersection of the capacity region of the two MACs. However, when the interference is weak or moderate, the capacity region is still unknown. The best inner bound of the capacity region is obtained in [4] by using superposition coding and joint decoding. A simplified form of the Han-Kobayashi region was given by Chong-Motani-Garg [6], [7]. Various outer bounds have been developed in [8]–[12]. Sato's outer bound in [8] is derived by allowing the receivers to cooperate. Carleial's outer bound in [9] is derived by decreasing the noise power. Kramer in [10] presented two outer bounds. The first is obtained by providing each receiver with just enough information to decode both messages. The second outer bound is obtained by reducing the IC to a degraded broadcast channel. Both of these two bounds dominate the bounds by Sato and Carleial. The recent outer bounds by Etkin, Wang, and Tse in [11] are also based on genie-aided methods, and they show that Han and Kobayashi's inner bound is within one bit or a factor of two of the capacity region. This result can also be established by the methods of Telatar and Tse [12]. We remark that neither of the bounds of [10] and [11] implies each other. But as a rule of thumb, our numerical results show that the bounds of [10] are better at low SNR while those of [11] are better at high SNR. The bounds of [12] are not amenable to numerical evaluation since the optimal distributions of the auxiliary random variables are unknown. None of the above outer bounds is known to be tight for the general Gaussian IC.

In this paper, we present a new outer bound on the capacity region of Gaussian ICs that improves on the bounds of [10], [11]. The new bounds are based on a genie-aided approach and a recently proposed extremal inequality [13]. Unlike the genie-aided method used in [10, Theorem 1], neither receiver is

required to decode the messages from the other transmitter. Based on this outer bound, we obtain new sum-rate capacity results (Theorem 2 and 3) for ICs satisfying some channel coefficient and power constraint conditions. We show that the sum-rate capacity can be achieved by treating the interference as noise when both the channel gain and the power are weak. We say that such channels have *noisy interference*. For this kind of noisy interference, the simple single-user transmission and detection strategy is sum-rate optimal. In Theorem 3, we show that for ICs with  $a > 1, 0 < b < 1$  and satisfying another condition, the sum-rate capacity is achieved by letting user 1 fully recover messages from user 2 first before decoding its own message, while user 2 only recovers its own messages.

This paper is organized as follows. In Section II, we present a new genie-aided outer bound and the resulting sum-rate capacity for certain Gaussian ICs. We prove these results in Section III. Numerical examples are given in Section IV, and Section V concludes the paper.

## II. MAIN RESULTS

### A. General outer bound

The following is a new outer bound on the capacity region of Gaussian ICs.

*Theorem 1:* If the rates  $(R_1, R_2)$  are achievable for IC( $a, b, P_1, P_2$ ) with  $0 < a < 1, 0 < b < 1$ , they must satisfy the following constraints (1)-(3) for  $\mu > 0, \frac{1+bP_1}{b+bP_1} \leq \eta_1 \leq \frac{1}{b}$  and  $a \leq \eta_2 \leq \frac{a+P_2}{1+aP_2}$ :

$$R_1 + \mu R_2 \leq \min_{\substack{\rho_i \in [0,1] \\ (\sigma_1^2, \sigma_2^2) \in \Sigma}} \frac{1}{2} \log \left( 1 + \frac{P_1^*}{\sigma_1^2} \right) - \frac{1}{2} \log (aP_2^* + 1 - \rho_1^2) + \frac{1}{2} \log \left( 1 + P_1 + aP_2 - \frac{(P_1 + \rho_1\sigma_1)^2}{P_1 + \sigma_1^2} \right) \\ + \frac{\mu}{2} \log \left( 1 + \frac{P_2^*}{\sigma_2^2} \right) - \frac{\mu}{2} \log (bP_1^* + 1 - \rho_2^2) + \frac{\mu}{2} \log \left( 1 + P_2 + bP_1 - \frac{(P_2 + \rho_2\sigma_2)^2}{P_2 + \sigma_2^2} \right), \quad (1)$$

$$R_1 + \eta_1 R_2 \leq \frac{1}{2} \log \left( 1 + \frac{b\eta_1 - 1}{b - b\eta_1} \right) - \frac{\eta_1}{2} \log \left( 1 + \frac{b\eta_1 - 1}{1 - \eta_1} \right) + \frac{\eta_1}{2} \log (1 + bP_1 + P_2), \quad (2)$$

$$R_1 + \eta_2 R_2 \leq \frac{1}{2} \log (1 + P_1 + aP_2) - \frac{1}{2} \log \left( 1 + \frac{a - \eta_2}{\eta_2 - 1} \right) + \frac{\eta_2}{2} \log \left( 1 + \frac{a - \eta_2}{a\eta_2 - a} \right), \quad (3)$$

where

$$\Sigma = \begin{cases} \left\{ (\sigma_1^2, \sigma_2^2) \mid \sigma_1^2 > 0, \quad 0 < \sigma_2^2 \leq \frac{1-\rho_1^2}{a} \right\}, & \text{if } \mu \geq 1, \\ \left\{ (\sigma_1^2, \sigma_2^2) \mid 0 < \sigma_1^2 \leq \frac{1-\rho_2^2}{b}, \quad \sigma_2^2 > 0 \right\}, & \text{if } \mu < 1, \end{cases} \quad (4)$$

and if  $\mu \geq 1$  we have

$$P_1^* = \begin{cases} P_1, & 0 < \sigma_1^2 \leq \left( \left( \frac{1}{\mu} - 1 \right) P_1 + \frac{1-\rho_2^2}{b\mu} \right)^+, \\ \frac{1-\rho_2^2-b\mu\sigma_1^2}{b\mu-b}, & \left( \left( \frac{1}{\mu} - 1 \right) P_1 + \frac{1-\rho_2^2}{b\mu} \right)^+ < \sigma_1^2 \leq \frac{1-\rho_2^2}{b\mu}, \\ 0, & \sigma_1^2 > \frac{1-\rho_2^2}{b\mu}, \end{cases} \quad (5)$$

$$P_2^* = P_2, \quad 0 < \sigma_2^2 \leq \frac{1-\rho_1^2}{a}, \quad (6)$$

where  $(x)^+ \triangleq \max\{x, 0\}$ , and if  $0 < \mu < 1$  we have

$$P_1^* = P_1, \quad 0 < \sigma_1^2 \leq \frac{1 - \rho_2^2}{b}, \quad (7)$$

$$P_2^* = \begin{cases} P_2, & 0 < \sigma_2^2 \leq \left( (\mu - 1) P_2 + \frac{\mu(1 - \rho_1^2)}{a} \right)^+, \\ \frac{\mu(1 - \rho_1^2) - a\sigma_2^2}{a - a\mu}, & \left( (\mu - 1) P_2 + \frac{\mu(1 - \rho_1^2)}{a} \right)^+ < \sigma_2^2 \leq \frac{\mu(1 - \rho_1^2)}{a}, \\ 0, & \sigma_2^2 > \frac{\mu(1 - \rho_1^2)}{a}. \end{cases} \quad (8)$$

Remark 1: The bounds (1)-(3) are obtained by providing different genie-aided signals to the receivers. There is overlap of the range of  $\mu$ ,  $\eta_1$ , and  $\eta_2$ , and none of the bounds uniformly dominates the other two bounds. Which one of them is active depends on the channel conditions and the rate pair.

Remark 2: Equations (2) and (3) are outer bounds for the capacity region of a Z-IC, and a Z-IC is equivalent to a degraded IC [5]. For such channels, it can be shown that (2) and (3) are the same as the outer bounds in [14]. For  $0 \leq \eta_1 \leq \frac{1+bP_1}{b+bP_1}$  and  $\eta_2 \geq \frac{a+aP_2}{1+aP_2}$ , the bounds in (2) and (3) are tight for a Z-IC (or degraded IC) because there is no power sharing between the transmitters. Consequently,  $\frac{1+bP_1}{b+bP_1}$  and  $\frac{a+aP_2}{1+aP_2}$  are the negative slopes of the tangent lines for the capacity region at the corner points.

Remark 3: The bounds in (2)-(3) turn out to be the same as the bounds in [10, Theorem 2]. We show this by proving that (3) is equivalent to [10, page 584, (37)-(38)] but with equalities rather than inequalities. Consider the rates

$$R_1 = \frac{1}{2} \log \left( 1 + \frac{P_1'}{P_2' + 1/a} \right) \quad (9)$$

$$R_2 = \frac{1}{2} \log (1 + P_2') \quad (10)$$

$$P_1' + P_2' = \frac{P_1}{a} + P_2 \quad (11)$$

for  $0 \leq P_1' \leq P_1$ . We rewrite (9) and (10) in the form of the weighted sum

$$R_1 + \alpha R_2 = \frac{1}{2} \log \left( 1 + \frac{P_1'}{P_2' + 1/a} \right) + \frac{\alpha}{2} \log (1 + P_2'). \quad (12)$$

Observe that (12) represents a line with slope  $\alpha$  where

$$\begin{aligned} \alpha &= -\frac{\partial R_1}{\partial R_2} \\ &= -\frac{\partial R_1}{\partial P_2'} \bigg/ \frac{\partial R_2}{\partial P_2'} \\ &= -\frac{\partial \log \left( 1 + \frac{P_1/a + P_2 - P_2'}{P_2' + 1/a} \right)}{\partial P_2'} \bigg/ \frac{\partial \log (1 + P_2')}{\partial P_2'} \\ &= \frac{a + aP_2'}{1 + aP_2'}. \end{aligned} \quad (13)$$

We thus obtain

$$P'_2 = \frac{a - \alpha}{a\alpha - a}. \quad (14)$$

Substituting (14) into (12), we have

$$R_1 + \alpha R_2 = \frac{1}{2} \log(1 + P_1 + aP_2) - \frac{1}{2} \log\left(1 + \frac{a - \alpha}{\alpha - 1}\right) + \frac{\alpha}{2} \log\left(1 + \frac{a - \alpha}{a\alpha - a}\right),$$

which is the same as (3). The relation  $a \leq \alpha \leq \frac{a+aP_2}{1+aP_2}$  follows from (13) and  $0 \leq P'_2 \leq P_2$ .

**Remark 4:** The bounds in [10, Theorem 2] are obtained by getting rid of one of the interference links to reduce the IC into a Z interference channel (or Z-IC, see [5]). Next, the proof in [10] allowed the transmitters to share their power, which further reduces the Z-IC into a degraded broadcast channel. Then the capacity region of this degraded broadcast channel is an outer bound for the capacity region of the original IC. The bounds in (2) and (3) are also obtained by reducing the IC to a Z-IC. Although we do not explicitly allow the transmitters to share their power, it is interesting that these bounds are equivalent to the bounds in [10, Theorem 2] with power sharing. In fact, a careful examination of our new derivation reveals that power sharing is implicitly assumed. For example, for the term  $h(X_1^n + Z_1^n) - \eta_1 h(\sqrt{b}X_1^n + Z_2^n)$  of (43) below, user 1 uses power  $P_1^* = \frac{b\eta_1 - 1}{b - b\eta_1} \leq P_1$ , while for the term  $\eta_1 h(Y_2^n)$  user 1 uses all the power  $P_1$ . This is equivalent to letting user 1 use the power  $P_1^*$  for both terms, and letting user 2 use a power that exceeds  $P_2$ . To see this, consider (43) below and write

$$\begin{aligned} n(R_1 + \eta_1 R_2) &\leq \frac{n}{2} \log(P_1^* + 1) - \frac{n\eta_1}{2} \log(bP_1^* + 1) + \frac{n\eta_1}{2} \log(1 + bP_1 + P_2) + n\epsilon \\ &= \frac{n}{2} \log(P_1^* + 1) - \frac{n\eta_1}{2} \log(bP_1^* + 1) + \frac{n\eta_1}{2} \log(1 + bP_1^* + P_2 + b(P_1 - P_1^*)) + n\epsilon \\ &= \frac{n}{2} \log(P'_1 + 1) - \frac{n\eta_1}{2} \log(bP'_1 + 1) + \frac{n\eta_1}{2} \log(1 + bP'_1 + P'_2) + n\epsilon, \end{aligned}$$

where  $P'_1 \triangleq P_1^*$ , and  $P'_2 \triangleq P_2 + b(P_1 - P_1^*)$ . Therefore, one can assume that user 2 uses extra power provided by user 1.

**Remark 5:** Theorem 1 improves [11, Theorem 3]. Specifically, for the three sum-rate bounds of [11, Theorem 3], the first bound can be obtained from (43) with  $P_1^* = P_1$  in (44). Therefore, the bound in (2) is tighter than the first sum-rate bound of [11, Theorem 3]. Similarly, the bound in (3) is tighter than the second sum-rate bound of [11, Theorem 3]. The third sum-rate bound in [11, Theorem 3] is a special case of (1) with  $\sigma_1^2 = \frac{1}{b}$ ,  $\sigma_2^2 = \frac{1}{a}$ ,  $\rho_1 = \rho_2 = 0$ .

**Remark 6:** Our outer bound is not always tighter than that of [11] for all rate points. The reason is that in [11, last two equations of (39)], different genie-aided signals are provided to the same receiver. Our

outer bound can also be improved in a similar and more general way by providing different genie-aided signals to the receivers. Specifically the starting point of the bound is

$$n(R_1 + \mu R_2) \leq \sum_{i=1}^k \lambda_i I(X_1^n; Y_1^n, U_i) + \sum_{j=1}^m \mu_j I(X_2^n; Y_2^n, W_j) + n\epsilon, \quad (15)$$

where  $\sum_{i=1}^k \lambda_i = 1$ ,  $\sum_{j=1}^m \mu_j = \mu$ ,  $\lambda_i > 0$ ,  $\mu_j > 0$ .

### B. Sum-rate capacity for noisy interference

The outer bound in Theorem 1 is in the form of an optimization problem. Four parameters  $\rho_1, \rho_2, \sigma_1^2, \sigma_2^2$  need to be optimized for different choices of the weights  $\mu, \eta_1, \eta_2$ . When  $\mu = 1$ , Theorem 1 leads directly to the following sum-rate capacity result.

*Theorem 2:* For the IC( $a, b, P_1, P_2$ ) satisfying

$$\sqrt{a}(bP_1 + 1) + \sqrt{b}(aP_2 + 1) \leq 1, \quad (16)$$

the sum-rate capacity is

$$C = \frac{1}{2} \log \left( 1 + \frac{P_1}{1 + aP_2} \right) + \frac{1}{2} \log \left( 1 + \frac{P_2}{1 + bP_1} \right). \quad (17)$$

*Remark 7:* The sum-rate capacity for a Z-IC with  $a = 0$ ,  $0 < b < 1$  is a special case of Theorem 2 since (16) is satisfied. The sum capacity is therefore given by (17).

Theorem 2 follows directly from Theorem 1 with  $\mu = 1$ . It is remarkable that a genie-aided bound is tight if (16) is satisfied since the genie provides extra signals to the receivers without increasing the rates. This situation is reminiscent of the recent capacity results for vector Gaussian broadcast channels (see [15]). Furthermore, the sum-rate capacity (17) is achieved by treating the interference as noise. We therefore refer to channels satisfying (16) as ICs with *noisy interference*. Note that (16) involves both channel gains  $a, b$  and both powers  $P_1$  and  $P_2$ . The constraint (16) implies that

$$\sqrt{a} + \sqrt{b} \leq 1. \quad (18)$$

Moreover, as shown in Fig. 1, the powers  $P_1$  and  $P_2$  must be inside the triangle defined by:

$$\begin{aligned} P_1 &\geq 0, \\ P_2 &\geq 0, \\ b\sqrt{a}P_1 + a\sqrt{b}P_2 &\leq 1 - \sqrt{a} - \sqrt{b}. \end{aligned} \quad (19)$$

These constraints can be considered as a counterpart of the IC with very strong interference [2] whose powers should be inside the rectangle defined in Fig. 2:

$$a > 1, b > 1,$$

$$0 \leq P_1 \leq a - 1,$$

$$0 \leq P_2 \leq b - 1.$$

The ICs with noisy interference and ICs with very strong interference are two extreme cases in terms of the decoding strategy to achieve the sum-rate capacity. In the former case, the sum-rate capacity is achieved by treating interference as noise, while in the latter case, the interference is decoded before, or together with, the intended messages.

For symmetric Gaussian ICs with  $a = b$  and  $P_1 = P_2$ , the conditions in (18) and (19) become

$$a = b \leq \frac{1}{4}, \quad (20)$$

$$P_1 = P_2 = P \leq \frac{\sqrt{a} - 2a}{2a^2}. \quad (21)$$

“Noisy interference” is therefore “weaker” than “weak interference” as defined in [5] and [16], namely

$$a \leq \frac{\sqrt{1+2P}-1}{2P} \text{ or}$$

$$P \leq \frac{1-2a}{a^2}. \quad (22)$$

Recall that [16] showed that for “weak interference” satisfying (22), treating interference as noise achieves larger sum rate than time-or frequency-division multiplexing (TDM/FDM), and [5] claimed that in “weak interference” the largest known achievable sum rate is achieved by treating the interference as noise.

### C. Capacity region corner point

*Theorem 3:* For an IC( $a, b, P_1, P_2$ ) with  $a > 1$ ,  $0 < b < 1$ , the sum-rate capacity is

$$C = \frac{1}{2} \log(1 + P_1) + \frac{1}{2} \log \left( 1 + \frac{P_2}{1 + bP_1} \right) \quad (23)$$

when the following condition holds

$$(1 - ab)P_1 \leq a - 1. \quad (24)$$

A similar result follows by swapping  $a$  and  $b$ , and  $P_1$  and  $P_2$ .

Under the constraint (24), we have the following inequality:

$$\frac{1}{2} \log \left( 1 + \frac{P_2}{1 + bP_1} \right) \leq \frac{1}{2} \log \left( 1 + \frac{aP_2}{1 + P_1} \right). \quad (25)$$

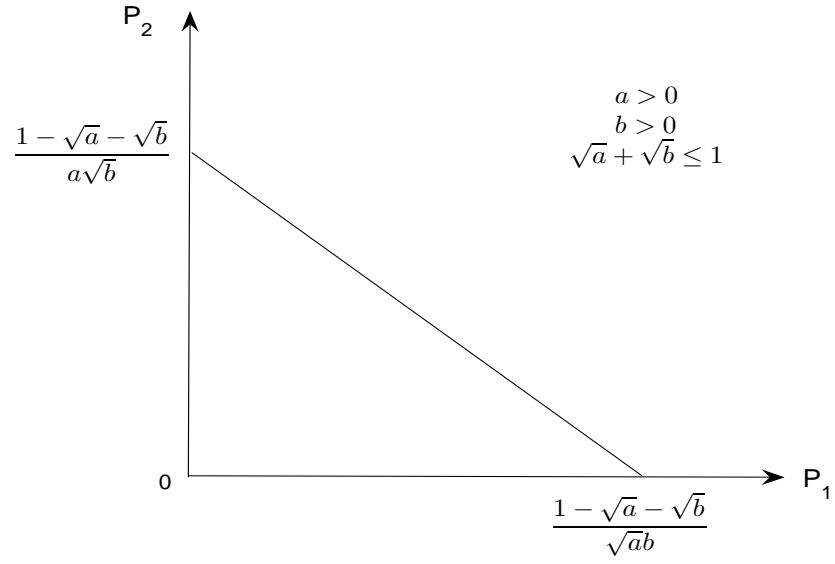


Fig. 1. Power region for the IC with noisy interference.

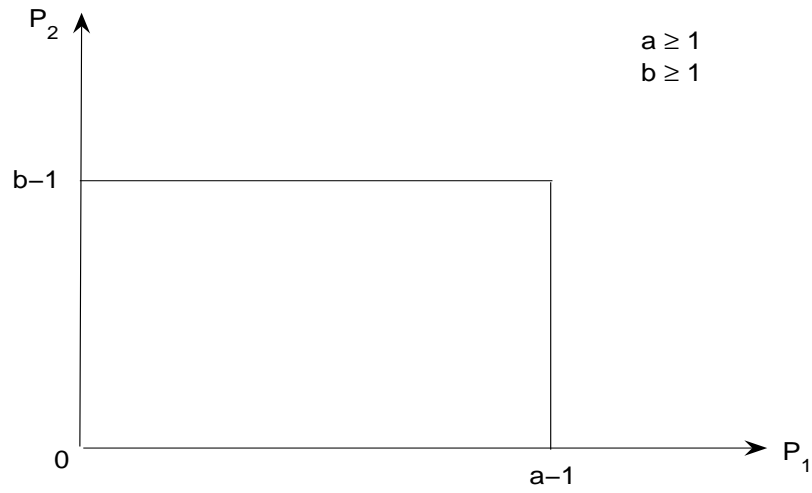


Fig. 2. Power region for the IC with very strong interference.



Therefore, the sum-rate capacity is achieved by a simple scheme: user 1 transmits at the maximum rate and user 2 transmits at the rate that both receivers can decode its message with single-user detection. Observe further that this rate pair permits  $R_2 = \frac{1}{2} \log \left( 1 + \frac{aP_2}{1+P_1} \right)$  when  $R_1$  reaches its maximum. Such a rate constraint was considered in [5, Theorem 1] which established a corner point of the capacity region. However it was pointed out in [16] that the proof in [5] was flawed. Theorem 3 shows that the rate pair of [16] is in fact a corner point of the capacity region when  $a > 1, 0 < b < 1$  and (24) is satisfied, and this rate pair achieves the sum-rate capacity.

The sum-rate capacity of the degraded IC ( $ab = 1, 0 < b < 1$ ) is a special case of Theorem 3. Besides this example, there are two other kinds of ICs to which Theorem 3 applies. The first case is  $ab > 1$ . In this case,  $P_1$  can be any positive value. The second case is  $ab < 1$  and  $P_1 \leq \frac{a-1}{1-ab}$ . For both cases, the signals from user 2 can be decoded first at both receivers.

#### D. State of the Art

We reiterate that both Theorems 2 and 3 are direct results of Theorem 1, and Theorem 1 is derived by having a genie provide extra information to the receivers. We summarize the sum-rate capacity for Gaussian ICs from Theorems 2 and 3 and previous results in [2]–[4]. In Fig. 3, four curves  $ab = 1$ ,  $a = 1$ ,  $b = 1$ , and  $\sqrt{a} + \sqrt{b} \leq 1$  divide channel gain plane into 7 regimes. The sum-rate capacity for each regime under certain power constraints is shown in Tab. I.

### III. PROOFS OF THE MAIN RESULTS

We introduce some notation. We write vectors and matrices by using a bold font (e.g.,  $\mathbf{X}$  and  $\mathbf{S}$ ). When useful we also write vectors with length  $n$  using the notation  $X^n$ . The  $i$ th entry of the vector  $\mathbf{X}$  (or  $X^n$ ) is denoted as  $X_i$ . Random variables are written as uppercase letters (e.g.,  $X_i$ ) and their realizations as the corresponding lowercase letter (e.g.,  $x_i$ ). We usually write probability densities and distributions as  $p(\mathbf{x})$  if the argument of  $p(\cdot)$  is a lowercase version of the random variable corresponding to this density or distribution. The notation  $h(\mathbf{X})$  and  $\text{Cov}(\mathbf{X})$  refers to the respective differential entropy and covariance matrix of  $\mathbf{X}$ . The notation of  $U|V = v$  and  $U|V$  denotes the random variable  $U$  conditioned on the event  $V = v$  and the random variable  $V$ , respectively.

The proof utilizes the extremal inequalities introduced in [13]. We present them below for completeness.

*Lemma 1:* [13, Theorem 1] For any  $\mu \geq 1$  and any positive semi-definite  $\mathbf{S}$ , a Gaussian  $\mathbf{X}$  is an optimal solution of the following optimization problem:

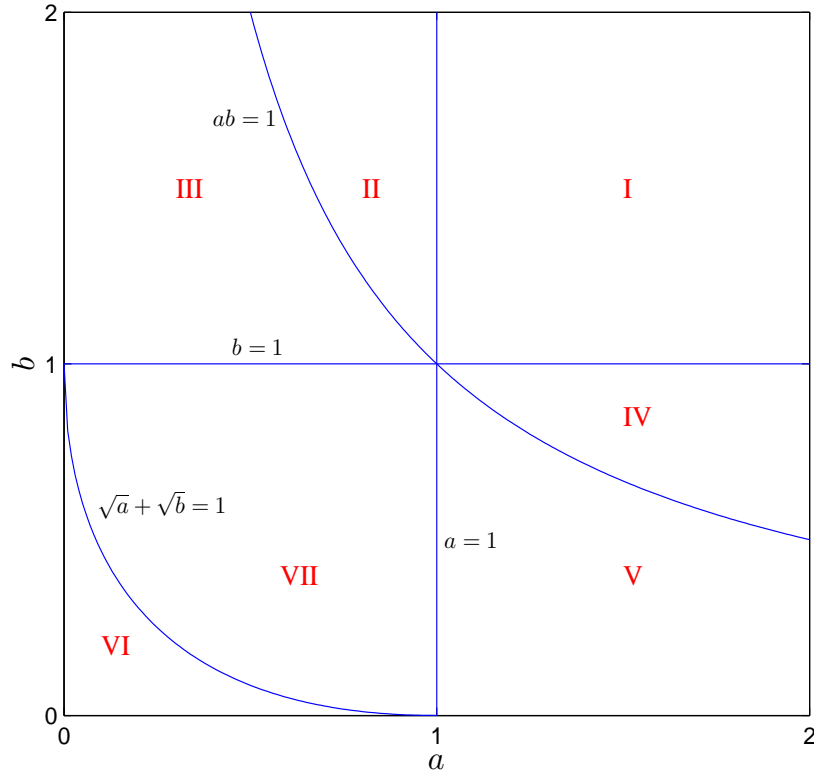


Fig. 3. Gaussian IC channel coefficient regimes for Tab. I

TABLE I  
SUM-RATE CAPACITY.

|     | $(a, b)$                                | $(P_1, P_2)$                                     | sum-rate capacity  |
|-----|---|--|--|
| I   | $a \geq 1, b \geq 1$                    | $P_1 > 0, P_2 > 0$                               | $\min \left\{ \begin{array}{l} \frac{1}{2} \log(1 + P_1) + \frac{1}{2} \log(1 + P_2) \\ \frac{1}{2} \log(1 + P_1 + aP_2) \\ \frac{1}{2} \log(1 + bP_1 + P_2) \end{array} \right\}$ |
| II  | $ab \geq 1, a \leq 1$                   | $P_1 > 0, P_2 > 0$                               | $\frac{1}{2} \log \left( 1 + \frac{P_1}{1+aP_2} \right) + \frac{1}{2} \log(1 + P_2)$   |
| III | $ab \leq 1, b \geq 1$                   | $P_1 > 0, P_2 \leq \frac{b-1}{1-ab}$             | same as above  |
| IV  | $ab \geq 1, b \leq 1$                   | $P_1 > 0, P_2 > 0$                               | $\frac{1}{2} \log(1 + P_1) + \frac{1}{2} \log \left( 1 + \frac{P_2}{1+bP_1} \right)$   |
| V   | $ab \leq 1, a \geq 1$                   | $P_1 \leq \frac{a-1}{1-ab}, P_2 > 0$             | same as above  |
| VI  | $\sqrt{a} + \sqrt{b} \leq 1$            | $\sqrt{a}(1 + bP_1) + \sqrt{b}(1 + aP_2) \leq 1$ | $\frac{1}{2} \log \left( 1 + \frac{P_1}{1+aP_2} \right) + \frac{1}{2} \log \left( 1 + \frac{P_2}{1+bP_1} \right)$  |
| VII | $\sqrt{a} + \sqrt{b} > 1, a < 1, b < 1$ | $P_1 > 0, P_2 > 0$                               | unknown  |

$$\begin{aligned} & \max_{p(\mathbf{x})} \quad h(\mathbf{X} + \mathbf{U}_1) - \mu h(\mathbf{X} + \mathbf{U}_2) \\ & \text{subject to} \quad \text{Cov}(\mathbf{X}) \preceq \mathbf{S}, \end{aligned}$$

where  $\mathbf{U}_1$  and  $\mathbf{U}_2$  are Gaussian vectors with strictly positive definite covariance matrices  $\mathbf{K}_1$  and  $\mathbf{K}_2$ , respectively, and the maximization is over all  $\mathbf{X}$  independent of  $\mathbf{U}_1$  and  $\mathbf{U}_2$ .

*Lemma 2:* [13, Corollary 4] For any real number  $\mu$  and any positive semi-definite  $\mathbf{S}$ , a Gaussian  $\mathbf{X}$  is an optimal solution of the following optimization problem:

$$\begin{aligned} & \max_{p(\mathbf{x})} \quad h(\mathbf{X} + \mathbf{U}_1) - \mu h(\mathbf{X} + \mathbf{U}_1 + \mathbf{U}) \\ & \text{subject to} \quad \text{Cov}(\mathbf{X}) \preceq \mathbf{S}, \end{aligned}$$

where  $\mathbf{U}_1$  and  $\mathbf{U}$  are two independent Gaussian vectors with strictly positive definite covariance matrices  $\mathbf{K}_1$  and  $\mathbf{K}$ , respectively, and the maximization is over all  $\mathbf{X}$  independent of  $\mathbf{U}_1$  and  $\mathbf{U}_2$ .

For example, consider the following optimization problem

$$\begin{aligned} & \max_{p(\mathbf{x})} \quad h(\mathbf{X} + \mathbf{U}_1) - \mu h(\mathbf{X} + \mathbf{U}_2) \\ & \text{subject to} \quad \frac{1}{n} \text{tr}(\mathbf{S}) \leq P, \quad \mathbf{S} = \mathcal{E}(\mathbf{X}\mathbf{X}^T), \end{aligned} \quad (26)$$

and suppose that  $\mathbf{S}^*$  is the optimal covariance matrix for  $\mathbf{X}$ . When  $\mu \geq 1$ , the problem (26) is equivalent to the problem of Lemma 1 with  $\mathbf{S}$  replaced by  $\mathbf{S}^*$ . Similarly, when  $\mu < 1$  the problem (26) is equivalent to the problem of Lemma 2 with  $\mathbf{S}$  replaced by  $\mathbf{S}^*$  and  $\mathbf{U}_2 = \mathbf{U}_1 + \mathbf{U}$ . Therefore a Gaussian  $\mathbf{X}$  is optimal for problem (26) in both cases. We further have the following two simple optimization results.

*Corollary 1:* The optimization problem of Lemma 1 with the matrix constraint replaced by the trace constraint (or the problem (26) with  $\mu \geq 1$ ) for the special case  $\text{Cov}(\mathbf{U}_i) = \sigma_i^2 \mathbf{I}$ ,  $i = 1, 2$ , has the solution  $\text{Cov}(\mathbf{X}) = P^* \mathbf{I}$ , where

$$P^* = \begin{cases} 0, & 0 < \sigma_2^2 < \mu \sigma_1^2 \\ \frac{\sigma_2^2 - \mu \sigma_1^2}{\mu - 1}, & \mu \sigma_1^2 \leq \sigma_2^2 < \mu \sigma_1^2 + (\mu - 1)P \\ P, & \sigma_2^2 \geq \mu \sigma_1^2 + (\mu - 1)P \end{cases} \quad (27)$$

Alternatively, we can write (27) as

$$P^* = \begin{cases} P, & 0 < \sigma_1^2 \leq \left( \frac{\sigma_2^2}{\mu} - \frac{\mu-1}{\mu} P \right)^+ \\ \frac{\sigma_2^2 - \mu \sigma_1^2}{\mu - 1}, & \left( \frac{\sigma_2^2}{\mu} - \frac{\mu-1}{\mu} P \right)^+ < \sigma_1^2 \leq \frac{\sigma_2^2}{\mu} \\ 0, & \sigma_1^2 > \frac{\sigma_2^2}{\mu} \end{cases} \quad (28)$$

*Corollary 2:* The optimization problem of Lemma 2 with the matrix constraint replaced by the trace constraint (or the problem of (26) with  $\mu < 1$  and  $\sigma_1^2 \leq \sigma_2^2$ ) for the special case  $\text{Cov}(\mathbf{U}_1) = \sigma_1^2 \mathbf{I}$ ,  $\text{Cov}(\mathbf{U}) = (\sigma_2^2 - \sigma_1^2) \mathbf{I}$ , where  $\sigma_1^2 \leq \sigma_2^2$ , has the solution  $\text{Cov}(\mathbf{X}) = P^* \mathbf{I}$  where

$$P^* = P. \quad (29)$$

*Proof:* Suppose the eigenvalue decomposition of  $\mathbf{S}$  is  $\mathbf{S} = \mathbf{Q} \mathbf{\Lambda} \mathbf{Q}^T$  and  $\mathbf{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_n)$ . Since Gaussian  $\mathbf{X}$  is optimal, we have

$$\begin{aligned} & h(\mathbf{X} + \mathbf{U}_1) - \mu h(\mathbf{X} + \mathbf{U}_2) \\ &= \frac{1}{2} \log [(2\pi e)^n |\mathbf{S} + \sigma_1^2 \mathbf{I}|] - \frac{\mu}{2} \log [(2\pi e)^n |\mathbf{S} + \sigma_2^2 \mathbf{I}|] \\ &= \frac{1}{2} \log |\mathbf{\Lambda} + \sigma_1^2 \mathbf{I}| - \frac{\mu}{2} \log |\mathbf{\Lambda} + \sigma_2^2 \mathbf{I}| + \frac{1-\mu}{2} \log (2\pi e)^n \\ &= \frac{1}{2} \sum_{i=1}^n \log (\lambda_i + \sigma_1^2) - \frac{\mu}{2} \sum_{i=1}^n \log (\lambda_i + \sigma_2^2) + \frac{1-\mu}{2} \log (2\pi e)^n \\ &\triangleq f(\mathbf{\Lambda}) \end{aligned}$$

By using the Lagrangian of  $f(\mathbf{\Lambda})$  with the constraint  $\sum_{i=1}^n \lambda_i = nP$ , it can be shown that the optimal  $\lambda_i$  is  $\lambda_i^* = P^*$  with  $P^*$  defined in (27)-(29). ■

Finally we need another lemma to prove our main results.

*Lemma 3:* Suppose that  $(U, V)$  is Gaussian with covariance matrix  $\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$ ,  $\sigma_1 > 0$ ,  $\sigma_2 > 0$ ,  $|\rho| < 1$ , and  $W$  is Gaussian with variance  $(1 - \rho^2) \sigma_1^2$ . If the discrete or continuous random variable  $X$  is independent of  $(U, V)$  and  $X$  is independent of  $W$ , then we have

$$h(X + U|V) = h(X + W) \quad (30)$$

*Proof:* We have

$$\begin{aligned} h(X + U|V) &= \int f_V(v) h(X + U|V = v) dv \\ &\stackrel{(a)}{=} \int f_V(v) h\left(X + W' + \frac{\rho\sigma_1}{\sigma_2} V \middle| V = v\right) dv \\ &\stackrel{(b)}{=} \int f_V(v) h(X + W') dv \\ &= h(X + W') \\ &= h(X + W) \end{aligned}$$

where  $W'$  is identically distributed as  $W$  but independent of  $(U, V)$ . (a) follows because  $(W' + \frac{\rho\sigma_1}{\sigma_2}V, V)$  has the same joint distribution as  $(U, V)$ , (b) follows because  $\frac{\rho\sigma_1}{\sigma_2}V$  becomes a constant when conditioned on  $V = v$ .  $\blacksquare$

Since  $U|V = v$  is also Gaussian distributed with mean value  $\frac{\rho\sigma_1}{\sigma_2}v$  and variance  $(1 - \rho^2)\sigma_1^2$ , Lemma 3 shows that  $U|V$  can be replaced by an equivalent Gaussian random variable with the same variance.

#### A. Proof of Theorem 1

Let  $N_1$  and  $N_2$  be two zero-mean Gaussian variables with variances  $\sigma_1^2$  and  $\sigma_2^2$  respectively, and set  $\mathcal{E}(N_1 Z_1) = \rho_1 \sigma_1$  and  $\mathcal{E}(N_2 Z_2) = \rho_2 \sigma_2$ . We further define  $N_1^n$  and  $N_2^n$  to be Gaussian vectors with  $n$  independent and identically distributed (i.i.d.) elements distributed as  $N_1$  and  $N_2$ , respectively.

Starting from Fano's inequality, we have that reliable communication requires

$$\begin{aligned}
& n(R_1 + \mu R_2) \\
& \leq I(X_1^n; Y_1^n) + \mu I(X_2^n; Y_2^n) + n\epsilon \\
& \leq I(X_1^n; Y_1^n, X_1^n + N_1^n) + \mu I(X_2^n; Y_2^n, X_2^n + N_2^n) + n\epsilon \\
& = I(X_1^n; X_1^n + N_1^n) + I(X_1^n; Y_1^n | X_1^n + N_1^n) + \mu I(X_2^n; X_2^n + N_2^n) + \mu I(X_2^n; Y_2^n | X_2^n + N_2^n) + n\epsilon \\
& = h(X_1^n + N_1^n) - h(N_1^n) + h(Y_1^n | X_1^n + N_1^n) - h(\sqrt{a}X_2^n + Z_1^n | N_1^n) + \mu h(X_2^n + Z_2^n) - \mu h(N_2^n) \\
& \quad + \mu h(Y_2^n | X_2^n + N_2^n) - \mu h(\sqrt{b}X_1^n + Z_2^n | N_2^n) + n\epsilon
\end{aligned} \tag{31}$$

where  $\epsilon \rightarrow 0$  as  $n \rightarrow \infty$ . For  $h(Y_1^n | X_1^n + N_1^n)$ , zero-mean Gaussian  $X_1^n$  and  $X_2^n$  are optimal, and we have

$$\begin{aligned}
\frac{1}{n}h(Y_1^n | X_1^n + N_1^n) & \leq \frac{1}{n} \sum_{i=1}^n h(Y_{1i} | X_{1i} + N_1) \\
& = \frac{1}{n} \sum_{i=1}^n (h(X_{1i} + \sqrt{a}X_{2i} + Z_1, X_{1i} + N_1) - h(X_{1i} + N_1)) \\
& = \frac{1}{2n} \sum_{i=1}^n \log \left[ 2\pi e \left( 1 + aP_{2i} + P_{1i} - \frac{(P_{1i} + \rho_1\sigma_1)^2}{P_{1i} + \sigma_1^2} \right) \right]
\end{aligned} \tag{32}$$

where  $P_{1i} = \mathcal{E}(X_{1i}^2)$  and  $P_{2i} = \mathcal{E}(X_{2i}^2)$ . Consider the function

$$f(p_1, p_2) = 1 + ap_2 + p_1 - \frac{(p_1 + \rho_1\sigma_1)^2}{p_1 + \sigma_1^2} \tag{33}$$

for which we compute

$$\frac{\partial f}{\partial p_1} = \frac{\sigma_1^2(\sigma_1 - \rho_1)^2}{(p_1 + \sigma_1^2)^2} \geq 0 \quad (34)$$

$$\frac{\partial f}{\partial p_2} = 1 \quad (35)$$

$$\frac{\partial^2 f}{\partial p_1^2} = -\frac{2\sigma_1^2(\sigma_1 - \rho_1)^2}{(p_1 + \sigma_1^2)^3} \leq 0 \quad (36)$$

$$\frac{\partial^2 f}{\partial p_2^2} = 0 \quad (37)$$

$$\frac{\partial^2 f}{\partial p_1 \partial p_2} = 0. \quad (38)$$

Since  $\log(x)$  is concave in  $x$  we have that the logarithm in (32) is concave in  $(P_{1i}, P_{2i})$ . We thus have

$$\begin{aligned} \frac{1}{n} h(Y_1^n | X_1^n + N_1^n) &\leq \frac{1}{2} \log \left[ 2\pi e \left( 1 + \frac{a}{n} \sum_{i=1}^n P_{2i} + \frac{1}{n} \sum_{i=1}^n P_{1i} - \frac{(\frac{1}{n} \sum_{i=1}^n P_{1i} + \rho_1 \sigma_1)^2}{\frac{1}{n} \sum_{i=1}^n P_{1i} + \sigma_1^2} \right) \right] \\ &\leq \frac{1}{2} \log \left[ 2\pi e \left( 1 + aP_2 + P_1 - \frac{(P_1 + \rho_1 \sigma_1)^2}{P_1 + \sigma_1^2} \right) \right] \end{aligned} \quad (39)$$

where the first inequality follows from Jensen's inequality, and the second inequality follows from the block power constraints  $\frac{1}{n} \sum_{j=1}^n P_{ij} \leq P_i$ ,  $i = 1, 2$ , and (34).

For the same reason, we have

$$\frac{1}{n} h(Y_2^n | X_2^n + N_2^n) \leq \frac{1}{2} \log \left[ 2\pi e \left( 1 + bP_1 + P_2 - \frac{(P_2 + \rho_2 \sigma_2)^2}{P_2 + \sigma_2^2} \right) \right]. \quad (40)$$

Let  $W_2' = Z_2 | N_2$ , then  $W_2'$  is Gaussian distributed with variance  $1 - \rho_2^2$ . Define a new Gaussian variable  $W_2$  with variance  $1 - \rho_2^2$ . From Lemma 3 and Corollaries 1 and 2 we have

$$\begin{aligned} &h(X_1^n + N_1^n) - \mu h(\sqrt{b}X_1^n + Z_2^n | N_2^n) \\ &= h(X_1^n + N_1^n) - \mu h(\sqrt{b}X_1^n + W_2^n) \\ &= h(X_1^n + N_1^n) - \mu h\left(X_1^n + \frac{W_2^n}{\sqrt{b}}\right) - \frac{n\mu}{2} \log b \\ &\leq \frac{n}{2} \log [2\pi e (P_1^* + \sigma_1^2)] - \frac{n\mu}{2} \log [2\pi e (bP_1^* + 1 - \rho_2^2)], \end{aligned} \quad (41)$$

where  $P_1^*$  is defined in (5) and (7). For the same reason, we have

$$\mu h(X_2^n + Z_2^n) - h(\sqrt{a}X_2^n + Z_1^n | N_1^n) \leq \frac{n\mu}{2} \log [2\pi e (P_2^* + \sigma_2^2)] - \frac{n}{2} \log [2\pi e (aP_2 + 1 - \rho_1^2)], \quad (42)$$

where  $P_2^*$  is defined in (6) and (8). From (31), (39)-(42) we obtain the rate constraint (1).

On the other hand, we have

$$\begin{aligned}
n(R_1 + \eta_1 R_2) &\leq I(X_1^n; Y_1^n) + \eta_1 I(X_2^n; Y_2^n) + n\epsilon \\
&\leq I(X_1^n; Y_1^n, X_2^n) + \eta_1 I(X_2^n; Y_2^n) + n\epsilon \\
&= I(X_1^n; Y_1^n | X_2^n) + \eta_1 I(X_2^n; Y_2^n) + n\epsilon \\
&= h(Y_1^n | X_2^n) - h(Y_1^n | X_1^n, X_2^n) + \eta_1 h(Y_2^n) - \eta_1 h(Y_2^n | X_2^n) + n\epsilon \\
&= h(X_1^n + Z_1^n) - \eta_1 h(\sqrt{b}X_1^n + Z_2^n) - h(Z_1^n) + \eta_1 h(Y_2^n) + n\epsilon \\
&\leq \frac{n}{2} \log(P_1^* + 1) - \frac{n\eta_1}{2} \log(bP_1^* + 1) + \frac{n\eta_1}{2} \log(1 + bP_1 + P_2) + n\epsilon, \quad (43)
\end{aligned}$$

where the last step follows by Corollaries 1 and 2. We further have

$$P_1^* = \begin{cases} P_1, & \eta_1 \leq \frac{1+bP_1}{b+bP_1} \\ \frac{b\eta_1-1}{b-b\eta_1}, & \frac{1+bP_1}{b+bP_1} \leq \eta_1 \leq \frac{1}{b} \\ 0, & \eta_1 \geq \frac{1}{b}. \end{cases} \quad (44)$$

Since the bounds in (43) when  $P_1^* = P_1$  and  $P_1^* = 0$  are redundant, we have

$$R_1 + \eta_1 R_2 \leq \frac{1}{2} \log \left( 1 + \frac{b\eta_1 - 1}{b - b\eta_1} \right) - \frac{\eta_1}{2} \log \left( 1 + \frac{b\eta_1 - 1}{1 - \eta_1} \right) + \frac{\eta_1}{2} \log(1 + bP_1 + P_2), \quad (45)$$

for  $\frac{1+bP_1}{b+bP_1} \leq \eta_1 \leq \frac{1}{b}$ , which is (2). We similarly obtain (3).

## B. Proof of Theorem 2

By choosing

$$\sigma_1^2 = \frac{1}{2b} \left\{ b(aP_2 + 1)^2 - a(bP_1 + 1)^2 + 1 + \sqrt{[b(aP_2 + 1)^2 - a(bP_1 + 1)^2 + 1]^2 - 4b(aP_2 + 1)^2} \right\} \quad (46)$$

$$\sigma_2^2 = \frac{1}{2a} \left\{ a(bP_1 + 1)^2 - b(aP_2 + 1)^2 + 1 + \sqrt{[a(bP_1 + 1)^2 - b(aP_2 + 1)^2 + 1]^2 - 4a(bP_1 + 1)^2} \right\} \quad (47)$$

$$\rho_1 = \sqrt{1 - a\sigma_2^2} \quad (48)$$

$$\rho_2 = \sqrt{1 - b\sigma_1^2}, \quad (49)$$

the bound (1) with  $\mu = 1$  is

$$R_1 + R_2 \leq \frac{1}{2} \log \left( 1 + \frac{P_1}{1 + aP_2} \right) + \frac{1}{2} \log \left( 1 + \frac{P_2}{1 + bP_1} \right). \quad (50)$$

By one can achieve equality in (50) by treating the interference as noise at both receivers.

In order that the choice of  $\sigma_1^2$ ,  $\sigma_2^2$ ,  $\rho_1$  and  $\rho_2$  be feasible, there must exist at least one pair  $(\sigma_1^2, \sigma_2^2)$  satisfying the following conditions:

$$\sigma_1^2 \geq 0, \quad \sigma_2^2 \geq 0, \quad \rho_1 \leq 1, \quad \rho_2 \leq 1.$$

Using (46)-(49), we thus require

$$[b(aP_2 + 1)^2 - a(bP_1 + 1)^2 + 1]^2 - 4b(aP_2 + 1)^2 \geq 0 \quad (51)$$

$$[a(bP_1 + 1)^2 - b(aP_2 + 1)^2 + 1]^2 - 4a(bP_1 + 1)^2 \geq 0 \quad (52)$$

$$b(aP_2 + 1)^2 - a(bP_1 + 1)^2 + 1 \geq 0 \quad (53)$$

$$a(bP_1 + 1)^2 - b(aP_2 + 1)^2 + 1 \geq 0. \quad (54)$$

From (51) we have one of the following three conditions

$$\sqrt{b}(aP_2 + 1) - \sqrt{a}(bP_1 + 1) \geq 1, \quad (55)$$

$$\sqrt{b}(aP_2 + 1) - \sqrt{a}(bP_1 + 1) \leq -1, \quad (56)$$

$$\sqrt{b}(aP_2 + 1) + \sqrt{a}(bP_1 + 1) \leq 1. \quad (57)$$

(52) gives the same constraints in (55)-(57). Since (53) and (54) exclude the possibilities (55) and (56), this leaves (57) which is precisely (16) in Theorem 2.

### C. Proof of Theorem 3

The proof of (2) requires only  $0 < b < 1$ . Therefore (2) is still valid when  $a > 1$ . Letting  $\eta_1 = 1$  and  $P_1^* = P_1$  in (43) and (44), we have the sum-rate capacity upper bound in (23). But (23) is achievable if (25) is true. To verify this, we let user 2 communicate at  $R_2 = \frac{1}{2} \log \left( 1 + \frac{P_2}{1+bP_1} \right)$ . From (25), user 1 can decode the message from user 2 before decoding its own messages. Then we obtained (24) and Theorem 3 is proved.

## IV. NUMERICAL EXAMPLES

A comparison of the outer bounds for a Gaussian IC is given in Fig. 4. Some part of the outer bound from Theorem 1 overlaps with Kramer's outer bound due to (2) and (3). Since this IC has noisy interference, the proposed outer bound coincides with the inner bound at the sum rate point.

The lower and upper bounds for the sum-rate capacity of the symmetric IC( $a = b, P_1 = P_2$ ) are shown in Figs. 5-8 for different power levels. For all of these cases, the upper bounds are tight up to point A. The bound in [11, Theorem 3] approaches to the bound in Theorem 1 when the power becomes large, but there is still a gap. Fig. 7 and 8 also provide a definitive answer to a question from [16, Fig. 2]: whether the sum-rate capacity of symmetric Gaussian IC is a decreasing function of  $a$ , or there exists a bump like the lower bound when  $a$  varies from 0 to 1. In Fig. 7 and 8, our proposed upper bound and



Sason's inner bound explicitly show that the sum capacity is not a monotone function of  $a$  (this result also follows by the bounds of [11]).

## V. CONCLUSIONS AND EXTENSIONS

We derived an outer bound for the capacity region of Gaussian ICs by a genie-aided method. From this outer bound, the sum-rate capacities for ICs that satisfy (16) or (24) are obtained.

We discuss in the following some possible extensions of the present work. One extension is already given in Remark 6 above. Another extension is to generalize the sum-rate capacity for a single noisy interference IC to that of parallel ICs, that occur in, for instance, orthogonal frequency division multiplexing (OFDM) systems. Finally, we note that the methods used in the paper can also be applied to obtain bounds for multiple input multiple output Gaussian ICs. We are currently developing such bounds.

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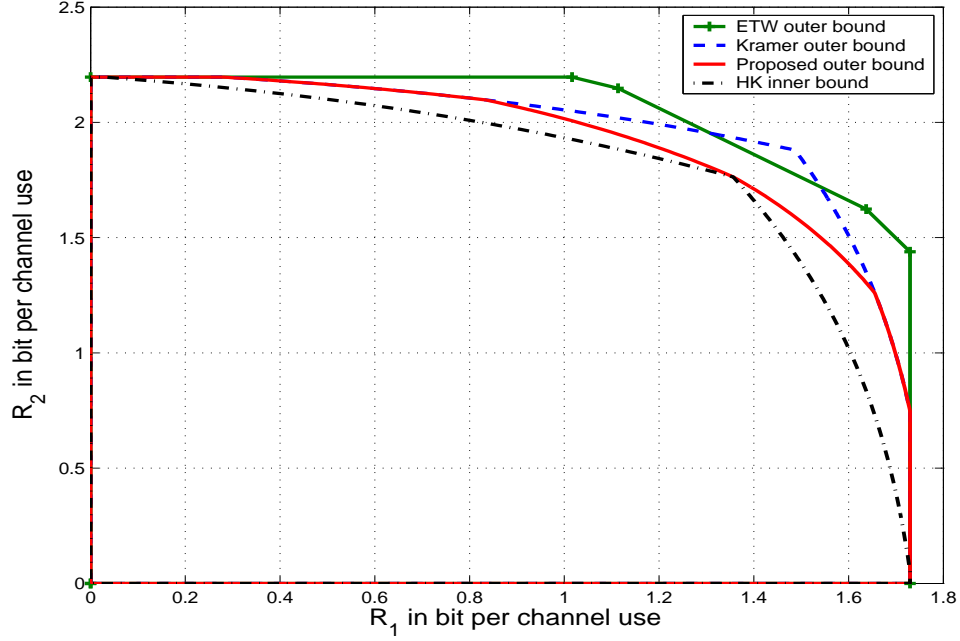


Fig. 4. Inner and outer bounds for the capacity region of Gaussian ICs with  $a = 0.09$ ,  $b = 0.04$ ,  $P_1 = 10$ ,  $P_2 = 20$ . The ETW bound is by Etkin, Tse and Wang in [11, Theorem 3]; the Kramer bound is from [10, Theorem 2]; the HK inner bound is based on [4] by Han and Kobayashi.

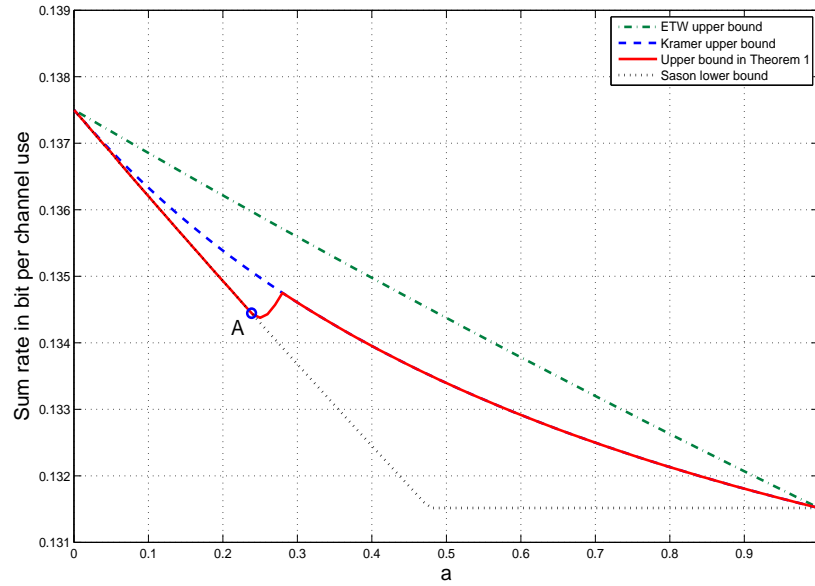


Fig. 5. Lower and upper bounds for the sum-rate capacity of symmetric Gaussian ICs with  $a = b$ ,  $P_1 = P_2 = 0.1$ . Sason's bound is an inner bound obtained from Han and Kobayashi's bound by a special time sharing scheme [16, Table I]. The channel gain at point A is  $a = 0.2385$ .

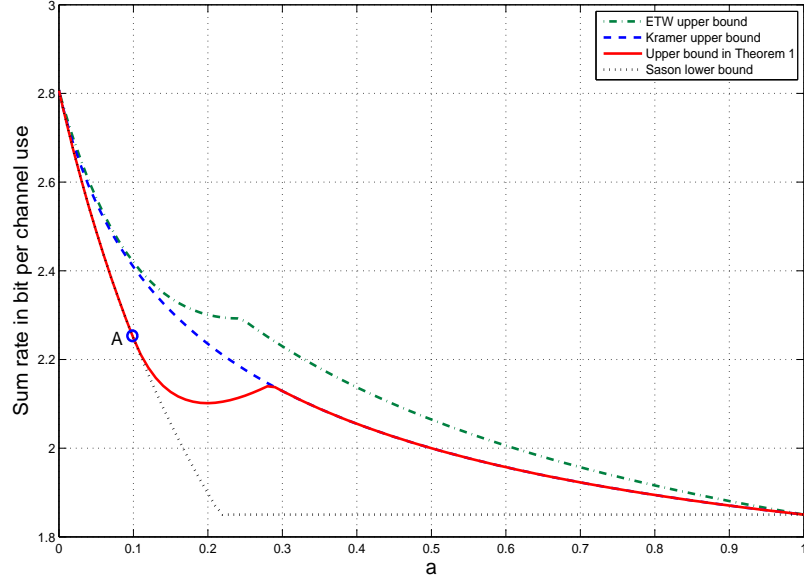


Fig. 6. Lower and upper bounds for the sum-rate capacity of symmetric Gaussian ICs with  $a = b$ ,  $P_1 = P_2 = 6$ . The channel gain at point  $A$  is  $a = 0.0987$ .

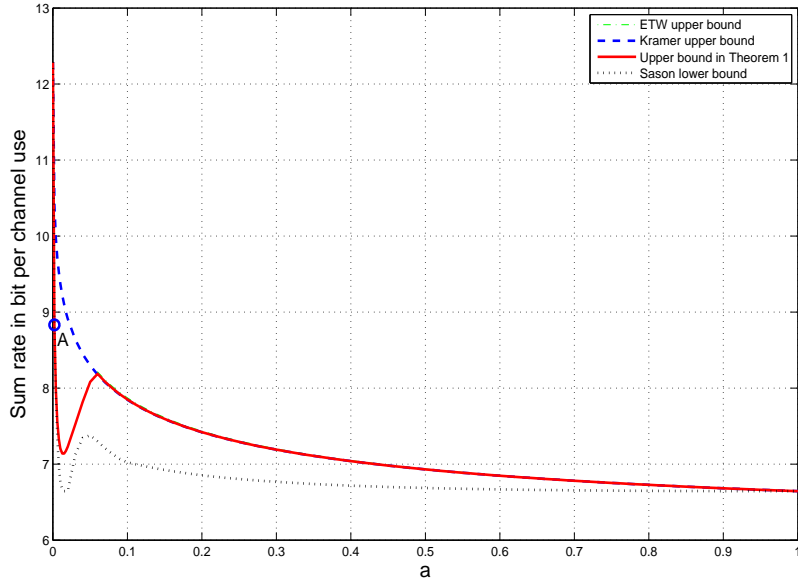


Fig. 7. Lower and upper bounds for the sum-rate capacity of symmetric Gaussian ICs with  $a = b$ ,  $P_1 = P_2 = 5000$ . The channel gain at point  $A$  is  $a = 0.002$ .

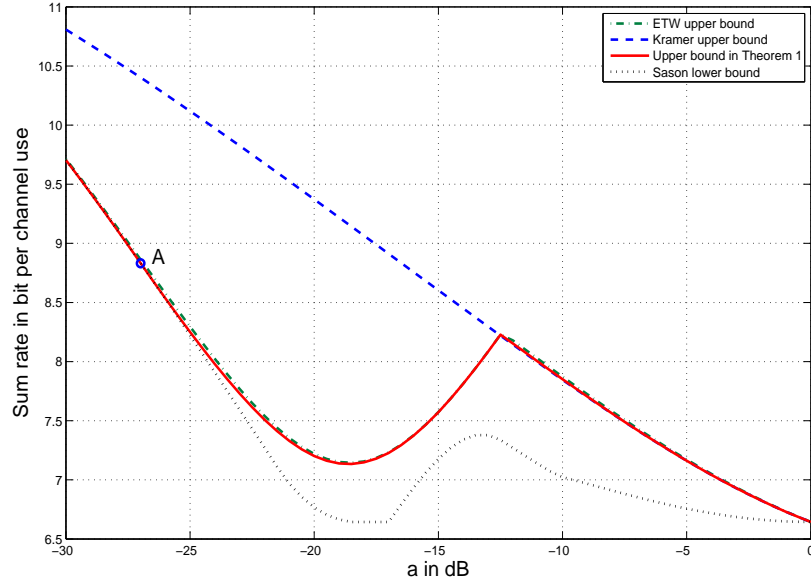


Fig. 8. Same as Fig. 7 with  $a$  replaced by  $10 \log_{10} a$ . The channel gain at point  $A$  is  $a = -26.99\text{dB}$ .

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